Chain Rule

-We can find the derivative of $y = \sin\left(x\right)$.

-We can also find $\frac{dy}{dx}$ .of $x^2 - 4$

-But what about $y = \sin\left(x^2 - 4\right)$

-For this we need the Chain Rule, one of the most widely used rules.

Example

The function $y = 6x - 10 = 2\left(3x - 5\right)$ is the composition of

$$y = 2u \qquad u = 3x - 5$$

How are the 3 derivatives related?

$$\frac{dy}{dx} = 6 \qquad .\frac{dy}{du} = 2. \qquad \frac{du}{dx} = 3$$

Since, $6 = 3 \bullet 2$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

Example

The polynomial $y = 9x^4 + 6x^2 + 1 = \left(3x^2 + 1\right)^2$ is the composition of $y = u^2$ and $u = 3x^2 + 1$.

-Calculating the derivatives we get

$$\frac{dy}{du} \bullet \frac{du}{dx} = 2u \bullet 6x$$

$$=2\left(3x^2+1\right)\bullet 6x=36x^3+12x$$

Also,

$$\frac{dy}{dx}=\frac{d}{dx}\left(9x^4+6x^2+1\right)$$

$$=36x^3+12x$$

Again,

$$\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$$

-The derivative of the composition function $f\left(g\left(x\right)\right)$ at x is the derivative of f at $g\left(x\right)$ times the derivative of g at x.

<u>The Chain Rule</u>

If f is differentiable at the point $u=g\left(x\right)$, and g is differentiable at x, then the composite function $\left(f\circ g\right)\left(x\right)=f\left(g\left(x\right)\right)$ is differentiable at x and

$$\left(f\circ g\right)'\left(x\right)=f'\left(g\left(x\right)\right)\circ g'\left(x\right)$$

-In Leibniz Notation if $y=f\left(u\right)$ and $u=g\left(x\right)$ then

$$\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$$

where $\frac{dy}{du}$ is evaluated at $u=g\left(x\right)$.

Example

-An object moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

-We know velocity is $\frac{dx}{dt}$

$x = \cos(u) \qquad u = t^2 + 1$

$$\frac{dx}{du} = -\sin u$$

$$\frac{du}{dt} = 2t$$

By the chain rule

$$\frac{dx}{dt} = \frac{dx}{du} \bullet \frac{du}{dt}$$

$$= -\sin(u) \bullet 2t$$
$$= -\sin(t^2 + 1) \bullet 2t$$
$$= -2t \sin(t^2 + 1)$$

Repeated Use of the Chain Rule

-We can use the chain rule 2 or more times to find a derivative as needed.

Example-3 linked chains

Find $g'(t)$ given $g(t) = \tan(5 - \sin 2t)$

$$g'(t) = \frac{d}{dx}\left(\tan\left(5 - \sin 2t\right)\right)$$

$$= \sec^2\left(5 - \sin 2t\right) \bullet \frac{d}{dt}\left(5 - \sin 2t\right)$$

$$= \sec^2\left(5 - \sin 2t\right) \bullet \left(0 - \cos 2t \bullet \frac{d}{dt}\left(2t\right)\right)$$

$$= \sec^2\left(5 - \sin 2t\right) \bullet \left(\cos 2t \bullet 2\right)$$

$$= -2\cos\left(2t\right)\sec^2\left(5 - \sin 2t\right)$$

Slopes of Parameterized Curves

Finding $\frac{dy}{dx}$ Parametrically

If all three derivatives exist $\frac{dy}{dx}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dx}{dt} \neq 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example

Find the line tangent to the right-hand hyperbola branch defined parametrically by

$$x = \sec t \qquad y = \tan t \qquad -\pi/2 < t < \pi/2$$

at the point $\left(\sqrt{2}, 1\right)$ where $t = \pi/4$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sec^2 t}{\sec t \tan t}$$

$$= \frac{\sec t}{\tan t} = \csc t$$

Setting $t = \pi/4$ gives

$$= \csc(\pi/4) = \sqrt{2}$$

The equation of the tangent line is

$$y - 1 = \sqrt{2}\left(x - \sqrt{2}\right)$$
$$y = \sqrt{2}x - 1$$

Power Chain Rule

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$$

Power Chain Rule

$$\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}$$

Example

-Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

$$\frac{dy}{dx} = 5\sin^4 x \bullet \frac{d}{dx}\sin x$$

$$= 5\sin^4 x \cos x$$

The tangent line has the slope

$$= 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}$$

Example

-Show that the slope of every line tangent to the curve $y = \dfrac{1}{(1-2x)^3}$ is positive.

$$\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3}$$

$$= -3(1-2x)^{-4} \bullet \frac{d}{dx}(1-2x)$$

$$= -3(1-2x)^{-4} \bullet (-2)$$

$$= \frac{6}{\left(1-2x\right)^4}$$

At any point $\left(x,y\right)$ on the curve $x \neq 1/2$ and the slope of the tangent line is

$\frac{dy}{dx} = \frac{6}{\left(1-2x\right)^4}$ where both the numerator and denominator as positive.